Coulomb Interactions in a Strong Magnetic Field. I. Adiabatic Case*

ROBERT GOLDMAN
Physics Department, Yale University, New Haven, Connecticut

AND

LUDWIG OSTER
Yale University Observatory, New Haven, Connecticut

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The line contour of cyclotron radiation from free electrons is derived quantum mechanically, including the effects of Coulomb interactions on the time-varying part of the wave functions ("adiabaticity") as a perturbation in the Hamiltonian. The computation is similar to Lindholm's theory of atomic lines. The results, line shifts and half-widths, i.e., collision cross sections, are given for a variety of physical conditions. The limitations of the treatment are discussed.

1. INTRODUCTION

THE purpose of this study is to investigate on a microscopic basis the effects of Coulomb interactions on the cyclotron line in a fully ionized plasma. The essential parameter is the collision cross section of the free electrons in the magnetic field with ions. The values most often used are either scattering cross sections derived from dc conductivity considerations, that do not account for the finite frequency of the radiation field, or are taken from the bremsstrahlung process, excluding effects of a magnetic field.

The method to be followed in this paper is to compute from the wave functions of free electrons in a magnetic field the matrix elements for dipole radiation, with the time-dependent part of the wave functions subject to Coulomb perturbations. The statistical superposition of these perturbations is carried out in the manner given by Lindholm[1] and leads directly to the expression for line shift and half-widths.

In general, the Coulomb interactions will affect the space-dependent part of the wave functions as well as the time-dependent part. In this paper, the effects on the space-dependent part of the wave functions are neglected. This restriction corresponds to the assumption of adiabaticity in the impact theory of line broadening. The "nonadiabatic" treatment involving changes of the spatial part of the wave functions is, for reasons obvious from the following sections, considerably more complicated, and will be discussed in a later communication. Since the two effects are separable, it was judged advisable to postpone the discussion of the nonadiabatic case.[2]

The work presented here accounts for a certain fraction of the cyclotron line's half-width and, for this matter, of the scattering cross section in a magnetic field, but it has an additional significance, because it predicts a line shift which does not follow from the nonadiabatic case. It is shown that this line shift should be observable under suitable laboratory conditions.

2. WAVE FUNCTIONS AND MATRIX ELEMENTS

The Schrödinger equation for an unperturbed electron in a magnetic field reads:

$$-\frac{\hbar^2}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2\psi=-i\hbar\frac{\partial\psi}{\partial t}=E\psi. \tag{1}$$

In Eq. (1), ψ is the wave function for the electron, E is the eigenenergy, e and m are the electron's charge and mass, c the velocity of light, $2\pi\hbar$ Planck's constant, $\mathbf{p}$ is the momentum operator. The vector potential $\mathbf{A}$ is related to the constant external magnetic field $\mathbf{H}$ by

$$\nabla\times\mathbf{A}=\mathbf{H}. \tag{2}$$

Equation (2) does, of course, not determine $\mathbf{A}$ uniquely in terms of a given magnetic field. A convenient choice for $\mathbf{A}$ consistent with Eq. (2) is to set

$$\mathbf{A}=\tfrac{1}{2}\mathbf{H}\times\mathbf{r}, \tag{3}$$

$\mathbf{r}$ being the radius vector to the field point from an arbitrary origin.

Let the z axis be chosen parallel to the magnetic field $\mathbf{H}$. The spatial part of the solutions to Eq. (1) is then found to be

$$\psi_{nks}=L^{-1/2}e^{ikz}(2\pi)^{-1/2}e^{i(n-s)\varphi}(2\gamma)^{+1/2}(n!s!)^{-1/2}\times\exp[-\rho/2]\rho^{(n-s)/2}Q_s{}^{n-s}(\rho). \tag{4}$$

The normalization length L is large and arbitrary. The parameter

$$\gamma=eH/2c\hbar \tag{5}$$

contains the magnetic field. ρ is defined by

$$\rho=\gamma r^2. \tag{6}$$

k is a continuous variable. s is integer and varies from

* Supported by the Aeronautical Research Laboratories of the Office of Aerospace Research, United States Air Force, Wright Patterson Air Force Base, Ohio.
[1] E. Lindholm, Ark. Mat. Astron. Fysik **28B**, No. 3 (1941).
[2] L. M. Tannenwald, Phys. Rev. **113**, 1396 (1959).

0 to ∞, $n-s$ varies from $-\infty$ to n.

$$Q_s^{n-s}(\rho)=(-)^s\sum_{\mu=0}^{s}\frac{(-)^\mu s!n!\rho^{s-\mu}}{\mu!(s-\mu)!(n-\mu)!} \tag{7}$$

is the generalized Laguerre polynomial.

The form of the solutions corresponds to Sokolov's solution[3] of the more general Dirac equation. The separate components of the solutions of Dirac's equation are themselves solutions of the corresponding Schrödinger equation. In this investigation, the Schrödinger equation will be used for mathematical convenience, limiting therefore the validity of the solutions to temperatures below, approximately, 10^{8}°K.

From Schrödinger's equation one finds the eigenenergies[4]

$$E_n=(n+\tfrac{1}{2})\hbar\omega_c+\hbar^2k^2/(2m), \tag{8}$$

where

$$\omega_c=eH/mc \tag{9}$$

is the cyclotron frequency. The matrix elements for dipole radiation are

$$|x|=\langle\psi_{n-1,s,k}^*|x|\psi_{n,s,k}\rangle=\tfrac{1}{2}(n/\gamma)^{1/2} \tag{10}$$

and

$$|y|=\langle\psi_{n-1,s,k}^*|y|\psi_{n,s,k}\rangle=\tfrac{1}{2}i(n/\gamma)^{1/2}. \tag{11}$$

The corresponding transition probability per unit time for decay by spontaneous emission is[5]

$$M_{\mathrm{sp}}=4e^2\omega_c^3[|x|^2+|y|^2][3\hbar c^3]^{-1}=\frac{4e^2\omega_c^2}{3\hbar c^3m}(n\hbar). \tag{12}$$

M_{sp} gives the total transition probability integrated over the line contour about the cyclotron frequency.[6]

3. INCLUSION OF COULOMB INTERACTIONS

So far, we have considered the wave function of electrons subject only to a constant external magnetic field. We now proceed to include the effects of interactions between the electron and positive ions in the two-particle approximation. The interaction adds a time and space varying part $H(\mathbf{r},t)$ to the Hamiltonian in Eq. (1) which now reads

$$\left[-\frac{\hbar^2}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2+H(\mathbf{r},t)\right]\psi=-i\hbar\frac{\partial\psi}{\partial t}. \tag{13}$$

Assuming that at the initial time ($t=-\infty$) there are no perturbations present, ψ is a solution of Eq. (1). The time-varying part can be written in the form $\exp[iEt/\hbar]$.

In order to facilitate comparison with similar work on atomic lines, we quote the alternate notation of Heisenberg for the wave function, i.e., we replace Schrödinger's ψ in Eq. (13) by the product of the time development matrix U and the wave function Ψ in the Heisenberg representation.[7] We then have

$$\left\{-\frac{\hbar^2}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2+H(\mathbf{r},t)\right\}U\Psi=-i\hbar\frac{\partial}{\partial t}(U\Psi). \tag{14}$$

The initial condition, i.e., that ψ at $t=-\infty$ is an eigensolution to the unperturbed Schrödinger equation, can then be written in the form

$$U_{ab}(t=-\infty)=\delta_{ab}, \tag{15}$$

where a and b are quantum numbers specifying the initial wave function. δ is the Kronecker symbol. Since the momentum in the z direction is proportional to the value of k, the specification of Eq. (4) introduces an infinite uncertainty in the position of the electron along the z axis.

For computing the contour of the cyclotron line due to interactions with heavy ions, the details of behavior during an encounter are of no interest, as it will be shown presently. Hence, the total change of the wave function in time can be replaced by the time average of the change caused by the interaction. For the same reason, in this approximation the variation of the wave functions in z direction need not be included.

Before we proceed to write down the solution of Eq. (14), we define the spectrally resolved intensity of spontaneous emission in terms of the wave functions ψ_i before and ψ_f after emission of a quantum with energy $\hbar\omega$:

$$I(\omega)=C\omega^4\left|\int_{-\infty}^{+\infty}\exp(-i\omega t)\int_{\mathbf{r}}\psi_f^*(\mathbf{r},t)e\mathbf{r}\psi_i(\mathbf{r},t)d\mathbf{r}dt\right|^2. \tag{16}$$

C is the normalization constant given, for instance, by Margenau and Lewis.[8] From the initial condition of Eq. (15) it follows that at time $t=-\infty$

$$U_f=\delta_{af}\delta_{bf},\quad U_i=\delta_{ai}\delta_{bi}. \tag{17}$$

The problem we are now faced with is to find the time variation of U_f and U_i caused by the perturbation described by Eq. (14).

The line broadening is then produced in the following way: Assume that a finite transition probability for dipole radiation exists only between states differing in quantum numbers by a certain value. The effect of collisions will be to spread the probability of finding a particle in a given initial or final state over a range of

[3] A. Sokolov, Suppl. Nuovo Cimento **3**, 743 (1956).

[4] R. J. Elliott and R. Loudon, J. Phys. Chem. Solids **15**, 196 (1960).

[5] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 262.

[6] We mention in passing that from the preceding formulas the line contour due to radiation damping can be derived. One finds

$$I(\omega)\propto\{(\omega-\omega_c)^2+[4e^2\omega_c^2n/(3mc^3)]^2\}^{-1}.$$

Under most conditions the half-width due to radiation damping is negligible compared with the half-width due to collisions. If this is not the case, the two quantities can be added in first approximation to an effective linewidth.

[7] P. W. Anderson, Phys. Rev. **76**, 647 (1949).

[8] H. Margenau and M. Lewis, Rev. Mod. Phys. **31**, 569 (1959).

quantum numbers. Since, in general, this spread will not be exactly the same for initial and final states, the overlap between the states, which develop in time from the initial and final states, changes as a function of time.

4. SOLUTIONS TO SCHRÖDINGER'S EQUATION

The problem of solving Eq. (14) in Heisenberg's notation, or the equivalent Schrödinger equation is exceedingly complex. Several simplifications will be introduced and discussed in turn.

Firstly, we represent the entire set of possible electron-ion interactions by the same electron wave function, but by varying ion positions **R**. The wave function of the electron can then be chosen as a function which is sharply peaked at the Larmor radius. The representation by sharply peaked functions results in a considerable mathematical simplification. In fact, specifying the energy in directions perpendicular and parallel to the magnetic field by quantum numbers n and k, respectively, is sufficient to determine the initial and final wave functions to be

$$\psi_i=\psi_{n,k,0},\quad \psi_f=\psi_{n-1,k,0} \tag{18}$$

in the notation of Eq. (4). Mathematically, Eq. (18) implies that

$$s=0 \tag{19}$$

or, in physical terms, that the origin is chosen coincident with the center of the orbit.[9] It is obvious from Eq. (7) that $s=0$ corresponds to a minimum spread in radial direction.

Secondly, we assume that the effect of the electron-ion interaction is to change the time varying part of the wave functions ψ_i and ψ_f without changing their spatial parts. In the following, this type of solution will be called "zero-order solutions." Then, ψ is separable in $\mathbf{r}$ and t, i.e.,

$$\psi_a(\mathbf{r},t)=\phi_a(\mathbf{r})\varphi_a(t). \tag{20}$$

Splitting the Hamiltonian $H(\mathbf{r},t)$ into a part H_0 containing the magnetic field and a part H_1 containing the Coulomb interaction, we have

$$[H_0+H_1(\mathbf{r},t)]\psi_a(\mathbf{r},t)=-i\hbar\partial\psi_a/\partial t. \tag{21}$$

Since there is no change in the spatial part of the wave function,

$$H_1(\mathbf{r},t)\phi_a(\mathbf{r})=\phi_a(\mathbf{r})H_1(\mathbf{r},t). \tag{22}$$

From Eq. (21) it follows that

$$E_a\phi_a(\mathbf{r})\varphi_a(t)+\phi_a(\mathbf{r})H(\mathbf{r},t)\varphi_a(t)=-i\hbar\phi_a(\mathbf{r})\varphi_a'(t), \tag{23}$$

where E_a denotes the eigenenergy introduced by Eq. (8). Multiplying by the complex conjugate of the spatial part of the wave function, and integrating over all space, we obtain

$$E_a\varphi_a(t)+\int\phi_a^*(\mathbf{r})\phi_a(\mathbf{r})H(\mathbf{r},t)\varphi_a(t)d\mathbf{r}=-i\hbar\varphi_a'(t). \tag{24}$$

Solving for $\varphi_a(t)$ results in

$$\varphi_a(t)=\exp\bigg[-(i\hbar)^{-1}\times\int_{-\infty}^{t}\bigg\{E_a+\int_{-\infty}^{+\infty}\phi_a^*(\mathbf{r})\phi_a(\mathbf{r})H(\mathbf{r},t)d\mathbf{r}\bigg\}dt\bigg]. \tag{25}$$

By taking Eq. (25) together with the previously discussed spatial part of the solution, for which Eq. (4) is to be inserted, we have a complete expression for the wave function.

It may be wise to note that a choice of

$$\phi_a(\mathbf{r})=\psi_{n,k,0}, \tag{26}$$

with the perturbing ion located at variable position R in the plane perpendicular to the magnetic field is equivalent to a choice of

$$\phi_a(\mathbf{r})=\psi_{n,k,s}, \tag{27}$$

with $s=0$, i.e., it is equivalent to placing the ion at the origin in the plane perpendicular to the magnetic field. Adopting $\phi_a(\mathbf{r})$ from Eq. (26) leads to the explicit specification of the problem. We have

$$\varphi_{n,k,0}(t)=\exp\bigg[-(i\hbar)^{-1}\int_0^t dt\times\bigg\{E_{n,k,0}+\int_{-\infty}^{+\infty}\phi_{n,k,0}^*\phi_{n,k,0}H(R,\mathbf{r},t)d\mathbf{r}\bigg\}\bigg]. \tag{28}$$

which is equivalent to Eq. (25).

5. PHYSICAL SIGNIFICANCE OF ZERO-ORDER SOLUTIONS

It was pointed out previously that making use of zero-order wave functions for the interaction corresponds to letting the wave function be constant in space. The physical significance of working with these solutions can be seen in the following way.

Firstly, compare the wave functions before and after the interaction, i.e., at times $-\infty$ and $+\infty$: The total energy of the electron cannot have changed. The implication is, therefore, that any energy loss due to radiation during the Coulomb interaction (bremsstrahlung) is neglected. This neglect can be justified whenever the Coulomb interaction is of a type, in which the ratio of the emitted photon energy $\hbar\omega$ to the total kinetic energy of electrons $\frac{1}{2}m\bar{v}^2$ is small, i.e., if

$$2\hbar\omega/m\bar{v}^2\ll 1. \tag{29}$$

[9] $(n-s)$ is the value of the generalized angular momentum.

The probability of emission of a photon with energy $\hbar\omega$ is greatest if, in the classical picture, the distance of closest approach

$$b \approx \bar{v}/\omega. \quad (30)$$

Therefore, only interactions with collision parameters

$$b \gg 2\hbar/m\bar{v} \quad (31)$$

can be treated according to this criterion. Under all physical conditions of interest, the number of interactions violating Eq. (31) is negligible.[10]

Secondly, use of zero-order functions also implies that the ratio of velocities parallel and perpendicular to the magnetic field is not changed by the Coulomb interaction. We therefore exclude any possible redistribution of kinetic energies by Coulomb interactions.

Thirdly, combining the two mentioned effects, one may say that the classical counterpart of assuming a spatially undisturbed wave function is to neglect a spatial change in the orbit of the particle due to the interaction. If there were no magnetic fields present, this orbit would be a straight line. The "straight-line approximation," however, is well justified even in most cases dealing with pure Coulomb interactions, where the magnetic field does not support the motion of the orbit's center along a straight line.[11]

A mathematical consequence of this "straight line" picture is the possibility of replacing the time coordinate by the z coordinate, i.e., the relation

$$dz = v_z dt, \quad (32)$$

with a constant velocity v_z along the magnetic field. Equation (32) is, of course, subject to the uncertainty principle, in the sense, that fixing the velocity v_z within certain limits results in corresponding limits for z.

6. COMPUTATION OF THE LINE CONTOUR BY LINDHOLM'S METHOD

Lindholm[1] has developed a theory for the broadening of spectral lines in the case of interactions producing a "random-phase change."[8] This type of interaction produces a certain change in the eigenfrequency of the oscillator, whereby in a large number of interactions the distribution of frequency modulations is random. In essence, the postulated randomness amounts to saying that the interactions are uncorrelated, and that their effects therefore are additive. The Debye shielding in a completely ionized plasma studied in this investigation does not affect the applicability of Lindholm's theory, since it will be shown in Sec. 6 that the effect of a finite cutoff length of the perturber's potential is negligible.

Lindholm begins his treatment by considering the expression for the intensity distribution in a spectral line, Eq. (16), which we specialize with the aid of Eq. (20),

$$I(\omega) = \text{const}\left|\langle\phi_a{}^*|\mathbf{r}|\phi_b\rangle \times \int_{-\infty}^{+\infty} \exp(-i\omega t)\varphi_a{}^*(t)\varphi_b(t)dt\right|^2. \quad (33)$$

For the time-varying part we take the solution (25). Equation (33) then becomes

$$I(\omega) = \text{const}\left|\langle\phi_a{}^*|\mathbf{r}|\phi_b\rangle \times \int_{-\infty}^{+\infty} \exp\left\{t\left[-i\omega + i\frac{E_B - E_A}{\hbar} + \Delta(a,b,t)\right]\right\}dt\right|^2, \quad (34)$$

where $\Delta(a,b,t)$ describes the effect of all the collisions and is represented by the following expression:

$$\Delta(a,b,t) = (i\hbar)^{-1}\int_{-\infty}^{t}\int_{-\infty}^{+\infty}(\phi_a{}^*\phi_a - \phi_b{}^*\phi_b) \times H(\mathbf{r},t')d\mathbf{r}dt'. \quad (35)$$

Alternately, if we fix the wave function, but let the ion's location be variable, we have for a particular ion located at $\mathbf{R}_i$.

$$\Delta(n,\mathbf{R}_i,t) = (i\hbar)^{-1}\int_{-\infty}^{t}\int_{-\infty}^{+\infty}[\phi_{n-1,k,0}{}^*\phi_{n-1,k,0} - \phi_{n,k,0}{}^*\phi_{n,k,0}] \times H(\mathbf{r},\mathbf{R}_i,t')dt'd\mathbf{r}, \quad (36)$$

with the Hamiltonian $H(\mathbf{r},\mathbf{R}_i,t')$ defined by

$$H(\mathbf{r},\mathbf{R}_i,t') = -\frac{Ze^2}{[(r-R_i)^2 + v_z{}^2(t'-t)^2]^{1/2}}, \quad t - t_0 \leq t' \leq t + t_0, \quad (37a)$$

$$H(\mathbf{r},\mathbf{R}_i,t') = 0, \quad \text{otherwise}. \quad (37b)$$

Z is the charge of the positive ion. At time t, the center of the electron's orbit passes through the ion location. t_0 is defined by

$$t_0 = [R_{\max}{}^2 - \rho_i{}^2]^{1/2}/v_z. \quad (38)$$

Let $\mathbf{k}$ denote the unit vector parallel to the magnetic field direction. Then

$$|\rho_i| = |\mathbf{R}_i \times \mathbf{k}|. \quad (39)$$

The finite value of t_0 takes the shielding of the ion potential into account. At present, there is no accurate theory available which predicts the correct value for $R_{\max}$ in a magnetic field. We shall use tentatively the Debye length

$$\lambda_D \approx (KT)^{1/2}/2\pi^{1/2}eN_e{}^{1/2} \quad (40)$$

as shielding distance, and verify that the numerical

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[10] L. Oster, Rev. Mod. Phys. **33**, 525 (1961); cf., Appendix H.
[11] See reference 10, Sec. 4.

results are insensitive to the choice of the cutoff parameter.

Equations (36) and (37) describe the effect of a single collision with an ion at position $\mathbf{R}_i$. Cumulative effects of many collisions are then obtained by summing over all collisions, i.e., by calculating

$$I(\omega)=\text{const}\left|\int_{-\infty}^{+\infty}\exp\left\{t\left[i\omega-i\frac{E_B-E_A}{\hbar}+\sum_i \Delta(N,\mathbf{R}_i,t)\right]\right\}dt\langle\phi_{n-1,k,0}^*|\mathbf{r}|\phi_{n,k,0}\rangle\right|^2. \quad (41)$$

Lindholm has carried out this calculation for a random distribution of perturbers, assuming that the time scale of the interaction is small compared with the lifetime of the states.[12] We shall verify below the applicability of this assumption by computing the relative magnitude of encounter time and linewidth.

It is not the scope of this paper to summarize the several steps in the transition from Eq. (41) to Lindholm's final form which reads:

$$I(\omega)d\omega=\frac{4e^2\omega^3}{3\pi mc^3}n\hbar\frac{u_1}{u_1^2+(\omega-\omega_c-u_2)^2}d\omega. \quad (42)$$

The line contour represented by Eq. (42) is very close to a Lorentzian form, except that in addition to the classical broadening represented by the parameter u_1 there occurs a shift of the resonance frequency

$$\omega_c=(E_B-E_A)/\hbar, \quad (43)$$

given by u_2.

The linewidth parameter u_1 reads

$$u_1=\nu\left[1-\int P(\mathbf{R})\cos\{\Delta(n,\mathbf{R})\}d\mathbf{R}\right], \quad (44)$$

with

$$\Delta(n,R)=-i\Delta(n,R,t=\infty). \quad (45)$$

$t=\infty$ represents the completed collision. The limits of integration in Eq. (44) are $R=0$ and $R=R_{\max}$.

For the finite value of $R_{\max}$ we take again the Debye shielding distance. It should be pointed out that the integration over the ion position $\mathbf{R}$ is extended only over the two components perpendicular to the direction of the magnetic field. Since we have neglected spatial correlations by the assumption of randomness, we are not concerned with the absolute value of the ion position in z direction.

The probability function $P(R)$ depends on the geometric probability of finding an ion at a distance R from the guiding center of the electron: We have

$$P(R)dR=2\pi RdR/\pi\lambda_D^2. \quad (46)$$

ν is the number of interactions per unit time. Since we have assumed complete randomness in the spatial distribution of ions, and did not allow for time changes of the velocity of the electrons, we find ν, again from obvious geometric considerations, to be

$$\nu=n_i\pi\lambda_D^2v_z. \quad (47)$$

n_i is the number of ions per cubic centimeter.

In a similar manner, u_2 which represents the shift of the resonance frequency in Eq. (42) is given by the expression[8]

$$u_2=\nu\int_{R=0}^{R=R_{\max}}P(\mathbf{R})\sin\{\Delta(n,\mathbf{R})\}d\mathbf{R}. \quad (48)$$

The problem of computing u_1 and u_2 reduces therefore to the evaluation of $\Delta(N,R,\infty)$ from Eq. (36). The mathematical details are summarized in Appendix A. The result reads

$$\Delta(n,R)\equiv\Delta_0=\frac{Ze^2}{\hbar vn}-\frac{Ze^2}{\hbar v}\left\{\frac{1}{\gamma\lambda_D^2+\gamma\lambda_D(\lambda_D^2-R^2)^{1/2}}-R^2\frac{2\lambda_D+(\lambda_D^2-R^2)^{1/2}}{2\gamma\lambda_D[\lambda_D^4+2\lambda_D^3(\lambda_D^2-R^2)^{1/2}+\lambda_D^2(\lambda_D^2-R^2)]}\right\},\quad \gamma R^2<n, \quad (49)$$

$$\Delta(n,R)=\Delta_0-Ze^2/\hbar vn,\quad \gamma R^2>n.$$

Equation (49) shows that the result, $\Delta(n,R)$ is discontinuous at $R=(n/\gamma)^{1/2}$. This fact appears to correspond to the physical situation that, for an ion position within the projection of the trajectory on the x-y-plane, the component of the force on the particle along the line joining ion position and the guiding center changes sign, whereas this sign change does not occur if the ion position is outside of the trajectory's projection. The first term in Eq. (49) outweighs the second term, which can be neglected in general. The error is of the order $(r_L/\lambda_D)^2$, r_L being the Larmor radius. In numbers, $\Delta(n,R)$ is positive for $R<(n/\gamma)^{1/2}$, and negative and much smaller for $R>(n/\gamma)^{1/2}$. In both cases, $|\Delta(n,R)|\ll 1$.

Using the fact that $|\Delta|\ll 1$, we expand $\sin\Delta$ and $\cos\Delta$ in the formulas for u_1 [half-width, Eq. (44)] and u_2 [line shift, Eq. (48)]. After some algebra, including again the neglect of terms of order $(r_L/\lambda_D)^2$ with regard to unity, we find

$$u_1=n_i\frac{\pi}{2}\frac{Z^2e^4}{\hbar^2}\frac{r_L^2}{v_zn^2}, \quad (50)$$

(n is the quantum number as before) and

$$u_2=n_i\frac{4\pi}{3}\frac{Zec}{H}. \quad (51)$$

[12] A more accurate quantum-mechanical definition using correlation functions is given by Margenau and Lewis, reference 8. See also the discussion in Sec. 7.

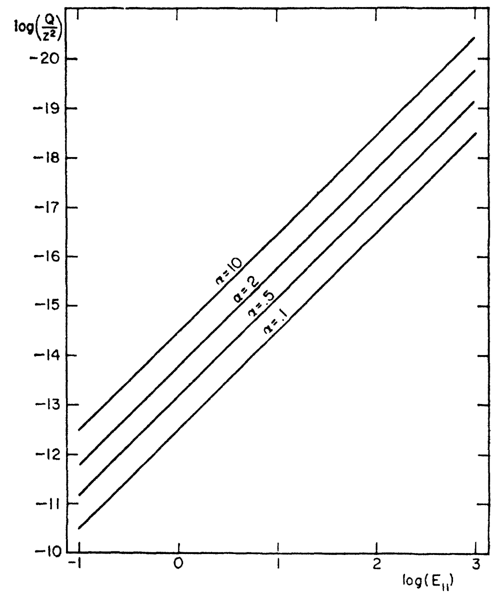

FIG. 1. Log(Q/Z^2) vs log$E_{\|}$ with $\alpha=E_{\perp}/E_{\|}$ as parameter. Q/Z^2 in cm^2, E in eV.

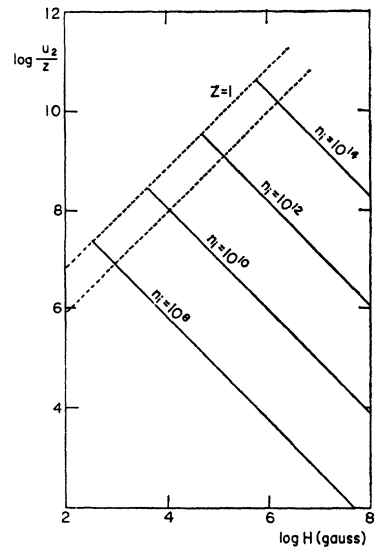

FIG. 2. Log(u_2/Z) vs log$H$ with n_i as parameter. u_2/Z in sec^{-1}, H in G. The dashed lines illustrate the locus for the occurrence of a line shift of 3% at low magnetic field intensities, corresponding to a nuclear charge $Z=1$ and $Z=10$, respectively. The solid lines may be extended to the upper left.

In terms of $E_{\perp}$ and $E_{\|}$, i.e., the kinetic energies of the electrons perpendicular and parallel to the magnetic field, we obtain instead of Eq. (50)

$$u_1=n_i\frac{\pi}{2}\frac{Z^2e^4v_z}{E_{\perp}E_{\|}}. \tag{52}$$

7. DISCUSSION OF THE LINE PARAMETERS

u_1, which represents the collision half-width of the cyclotron line depends on the ion density n_i, the ionic charge Z, and the electron energies $E_{\perp}$ and $E_{\|}$. It neither depends on the magnetic field nor on the cutoff distance λ_D. The independence of λ_D is plausible since we assume throughout this paper that the electron motion is determined by the magnetic field rather than the Coulomb interaction. The independence of the magnetic field is derived from the compensation of the strength of the interaction and its probability. This can be seen from Eqs. (44) and (49) which state that the strength of the interaction is proportional to $[\Delta n,R)]^2\propto r_L^{-2}$ after expanding $\cos[\Delta(n,R)]$, whereas the probability of a single interaction is proportional to r_L^2. Therefore, in averaging over all interactions (of particles with the same parallel and perpendicular energies), the magnetic field vanishes from the collision half-width.

From Eq. (52) we obtain geometric cross-sections Q by the conventional definition

$$u_1=Qv_zn_i. \tag{53}$$

Figure 1 shows a plot of log(Q/Z^2) vs log$E_{\|}$ with

$$\alpha=E_{\perp}/E_{\|} \tag{54}$$

as parameter.

The line shift, which is not obtained from the simple collision theory of the Lorentz-type, depends on the ion density and the magnetic field, but not on the electron's energy. Since the nonadiabatic spatial variations of the wave functions do not lead to a line shift, u_2 is the *final result.* We have plotted in Fig. 2 log(u_2/Z) vs log$H$ with n_i as parameter. The shift increases with increasing ion density and decreasing magnetic field, as long as the basic assumption

$$r_L^2\ll\lambda_D^2 \tag{55}$$

is satisfied. For a combination of $n_i=10^{10}$ cm^{-3}, $H=10^3$G, the line shift is about 3% of the gyrofrequency. Hence, it should be observable under suitable experimental conditions.

8. LINEWIDTH FOR A MAXWELLIAN ASSEMBLY OF ELECTRONS

For practical applications, the average over the linewidth parameter u_1 with regard to a distribution function of electron energies is needed. As mentioned in the Introduction, the line contour due to adiabatic interactions by itself is not too significant for experimental applications since the observed line shape for cyclotron radiation is due to the combination of adiabatic and nonadiabatic interactions with the addition of Doppler effects. However, the calculation of the line contour does provide an effective collision half-width for adiabatic interactions, and a measure of deviations from the Lorentzian shape for the collision broadened line, which then may be folded with the Doppler profile and, of course, the nonadiabatic effects.

TABLE I. Line contour for a Maxwell-Boltzmann distribution. $a=n_i\pi Z^2e^4(mKT)^{-1}(m/2KT)^{1/2}$. Column III is explained in the text.

$\Delta\omega/a$	$\mathfrak{g}(\Delta\omega)$	III
0	1.00	1.00
0.1	0.93	0.97
0.2	0.77	0.87
0.5	0.50	0.50
1.0	0.29	0.20
2.0	0.15	0.06
5.0	0.05	0.01

Using u_1 from Eq. (52) and the expression for the intensity given by Eq. (42), we obtain by integrating over the distribution function of electron velocities:

$$I(\omega)=-\frac{2}{3\pi}\frac{e^2\omega^2}{mc^3}\int_{-\infty}^{+\infty}P(E_z)\times\int_0^\infty E_\perp P(E_\perp)dE_\perp\left[-\frac{2n_i\pi Z^2e^4}{mv_zE_\perp}\right]\times\left[\left(\frac{n_i\pi Z^2e^4}{mv_zE_\perp}\right)^2+(\omega_c+u_2-\omega)^2\right]^{-1}d(E_z^{1/2}), \quad (57)$$

writing $P(E_z)d(E_z^{1/2})$ for the relative probability of finding a single particle with a velocity between $E_z^{1/2}$ and $E_z^{1/2}+d(E_z^{1/2})$, and $P(E_\perp)dE_\perp$ for the relative probability of finding a single particle with an energy between $E_\perp$ and $E_\perp+dE_\perp$.

Assuming a Maxwell-Boltzmann distribution for the electrons we have

$$P(E_z)d(E_z^{1/2})=\exp[-E_z/KT]d(E_z^{1/2})/(\pi KT)^{1/2} \quad (59)$$

and

$$P(E_\perp)dE_\perp=\exp[-E_\perp/KT]d(E_\perp/KT). \quad (60)$$

Letting

$$\Delta\omega=\omega_c+u_2-\omega=G(n_i\pi Z^2e^4/mv_zKT), \quad (61)$$

Eq. (56) becomes

$$I(\omega)=\frac{4e^2\omega^2}{3\pi mc^3}\times\int_{-\infty}^{+\infty}\frac{\exp[-mv_z^2/2KT]d(E_z^{1/2})\,(KT)^2f(G)}{(\pi KT)^{1/2}(n_i\pi Z^2e^4/mv_z)}, \quad (62)$$

where

$$f(G)=\frac{1}{G^2}+\frac{1}{G^3}\left\{\sin\left(\frac{1}{G}\right)\mathrm{ci}\left(\frac{1}{G}\right)+\cos\left(\frac{1}{G}\right)\mathrm{si}\left(\frac{1}{G}\right)\right\}, \quad (63)$$

with

$$\mathrm{si}\left(\frac{1}{G}\right)=\int_{1/G}^\infty\frac{\sin x}{x}dx,\quad \mathrm{ci}\left(\frac{1}{G}\right)=\int_{1/G}^\infty\frac{\cos x}{x}dx. \quad (64)$$

TABLE II. Line shift and width for typical values of ion density N_i (cm^{-3}), temperature T(°K), and field strength H(G).

$N_i=10^{10}$; $H=10^3$	u_1/ω_c	u_{br}/ω_c	u_{sc}/ω_c	
$T=10^4$	1.4	21	25	$[10^{-6}]$
$T=10^5$	4.4	97	103	$[10^{-8}]$
$T=10^6$	1.4	30	41	$[10^{-9}]$
$N_i=10^{10}$; $H=10^6$				
$T=10^4$	1.4	6.4	25	$[10^{-9}]$
$T=10^5$	4.4	43	103	$[10^{-11}]$
$T=10^6$	1.4	23	41	$[10^{-12}]$
$N_i=10^{14}$; $H=10^6$				
$T=10^4$	1.4	6.4	14	$[10^{-5}]$
$T=10^5$	4.4	43	69	$[10^{-7}]$
$T=10^6$	1.4	23	30	$[10^{-8}]$

On letting

$$v_z=(2KT/m)^{1/2}x^{1/2}, \quad (65)$$

we obtain

$$I(\omega)=\frac{8\sqrt{2}\omega^2(KT)^{5/2}}{3\pi^{5/2}e^2Z^2n_im^{1/2}c^3}\mathfrak{g}(\Delta\omega), \quad (66)$$

where

$$\mathfrak{g}(\Delta\omega)=\int_0^\infty\tfrac{1}{2}e^{-x}f\left(\frac{\Delta\omega x^{1/2}}{(n_i\pi Z^2e^4/mKT)(m/2KT)^{1/2}}\right)dx. \quad (67)$$

A simple numerical calculation yields for $\mathfrak{g}(\Delta\omega)$ the approximate values in Table I. Normalizing the half-width of the Lorentzian line to

$$\Delta\omega\left[\frac{n_i\pi Z^2e^4}{mKT}\left(\frac{m}{2KT}\right)^{1/2}\right]^{-1}=0.50 \quad (68)$$

with center intensity one, we obtain column III of Table I.

From Table I, it follows that the half-width of the adiabatic part of the cyclotron line in the case of a Maxwell-Boltzmann distribution of electrons is

$$0.5n_i\pi Z^2e^4(m/2KT)^{1/2}(mKT)^{-1}. \quad (69)$$

The effect of a Maxwellian spread of velocities is the shifting of a part of the intensity distribution from the line core to the wings.

For a rough estimate of orders of magnitude, some representative values of u_1 in units of the gyrofrequency ω_c are given in Table II. In computing these values we have assumed equal partition of $E_\perp$ and $E_{||}$. For $E_\perp$ and $E_{||}$ we take the equilibrium values

$$E_{||}=\tfrac{1}{2}KT,\quad E_\perp=KT. \quad (70)$$

Also included in Table I are values u_{sc} of the half-width computed with the aid of momentum-transfer cross sections, i.e., from scattering processes of the type considered in dc conductivities.[13] The values u_{br} are derived in a similar manner making use of "bremsstrahlung cross sections." Here the absorption coefficient for a

[13] R. S. Cohen, L. Spitzer, and P. Routly, Phys. Rev. **80**, 230 (1950); L. Spitzer and R. Harm, *ibid.* **89**, 977 (1953).

bremsstrahlung process is written in terms of a cross section.[14] with the frequency ω fixed at the gyrofrequency. u_{br} and u_{sc} are derived neglecting adiabatic effects.

It should also be noted that the adiabatic broadening because of its $E_\perp^{-1}v_z^{-1}$ dependence may become comparable to the more significant nonadiabatic broadening for distribution functions for which the average energy in the direction parallel to the field is very different from the average energy in the direction perpendicular to the field. This is the case in many thermofusion plasmas.

APPENDIX A

We summarize here the mathematical computation of $\Delta(N,R,\infty)$ used in the development of Sec. 5.

According to Eqs. (30) and (37),

$$\Delta(N,R,\infty)=(i\hbar)^{-1}\int_{t'-t=-t_0}^{t'-t=t_0}\int_{-\infty}^{+\infty}(\phi_{n-1,k,0}{}^*\phi_{n-1,k,0}-\phi_{n,k,0}{}^*\phi_{n,k,0})\frac{-Ze^2}{[(\mathbf{r}-\mathbf{R}_i)^2+v_z^2(t'-t)^2]^{1/2}}dt'd\mathbf{r}. \tag{A1}$$

The variation of $\phi_{n,k,0}$ in the direction along the magnetic field is given by

$$\phi_{n,k,0}\sim e^{ikz}/L^{1/2}, \tag{A2}$$

according to Eq. (4). By the uncertainty principle we have

$$\Delta p_z\Delta_z\sim\hbar. \tag{A3}$$

Δp_z is the uncertainty of the momentum in z direction, Δz the uncertainty of the position in z direction. It follows that $\Delta v_z\Delta z\approx\hbar/m$. Δv_z is the uncertainty of the velocity component in the z direction.

Consider an electron wave packet localized to within Δz at the beginning of the transit of an electron through the Debye sphere surrounding the ion. During the transit, there is the following spread in the wave pocket:

$$\Delta v_z(\lambda_D/v_z)=\frac{\hbar}{m}\frac{\lambda_D}{v_z}(\Delta z)^{-1}. \tag{A4}$$

Minimizing after transits, the width of the wave packet is of the order

$$\Delta z+\Delta v_z(\lambda_D/v_z)\approx 2\left(\frac{\hbar}{m}\frac{\lambda_D}{v_z}\right)^{1/2}. \tag{A5}$$

If this width is small with regard to the Debye-length λ_D, the electron may certainly be regarded as a particle during the interaction, at least, as far as the z direction is concerned. Then, $\exp[ikz]L^{-1/2}$ can be replaced by

$$[\delta(z-v_z(t'-t))]^{1/2}. \tag{A6}$$

From the condition

$$2(\hbar\lambda_D/mv_z)^{1/2}\ll\lambda_D \tag{A7}$$

we obtain in numbers

$$10^{-3}(n^{1/2}/T)^{1/2}\ll 1, \tag{A8}$$

with $v_z\approx(KT/m)^{1/2}$ and

$$\lambda_D=7(T/N_e)^{1/2}. \tag{A9}$$

The inequality (A7) holds for all "independent particle plasmas".[15] Equation (A1) then becomes [cf. Eqs. (32) and (A6)]:

$$\Delta(N,R,\infty)$$
$$=(i\hbar)^{-1}\int_{t'-t=-t_0}^{t'-t=t_0}\int_{x=0}^{\infty}\int_{z=-\infty}^{\infty}\int_{\theta=0}^{2\pi}\delta[z-v_z(t'-t)]\times\left(\frac{e^{-x}x^{n-1}}{(n-1)!2\pi}-\frac{e^{-x}x^n}{n!2\pi}\right)\times\left(\frac{-Ze^2}{[(\mathbf{r}-\mathbf{R}_i)^2+v_z^2(t'-t)^2]^{1/2}}\right)dt'dxdzd\theta, \tag{A10}$$

or, since the integrand is independent of z,

$$\Delta(N,R,\infty)$$
$$=(i\hbar)^{-1}\int_{t'-t=-t_0}^{t'-t=t_0}\int_{x=0}^{\infty}\int_{\theta=0}^{2\pi}\frac{e^{-x}}{2\pi}\left(\frac{x^{n-1}}{(n-1)!}-\frac{x^n}{n!}\right)\times\left(\frac{-Ze}{[(\mathbf{r}-\mathbf{R}_i)^2+v_z^2(t'-t)^2]^{1/2}}\right)dt'dxd\theta. \tag{A11}$$

Recalling the fact that

$$v_z(t'-t)=z, \tag{A12}$$

the time integration can be transformed back into an integration over z:

$$\Delta(n,\rho_i)=\Delta(n,R)=-i\Delta(N,R,\infty)$$
$$=(\hbar v_z)^{-1}\int_{-(\lambda_D^2-\rho_i^2)}^{(\lambda_D^2-\rho_i^2)}\int_{x=0}^{\infty}\int_{\theta=0}^{2\pi}\frac{e^{-x}}{2\pi}\left(\frac{x^n}{n!}-\frac{x^{n-1}}{(n-1)!}\right)\times\left(\frac{-Ze^2}{[(\mathbf{r}-\rho_i)^2+z^2]^{1/2}}\right)dzdxd\theta. \tag{A13}$$

The limits on $x=\gamma r^2$ [cf. Eq. (5)] are not critical, since the wave functions are sharply peaked at $r_L\ll\lambda_D$.

[14] See reference 10, Appendix B.

[15] J. L. Delcroix, *Introduction to the Theory of Ionized Gases* (Interscience Publishers, Inc., New York, 1961), p. 108.

The integration can be readily performed and yields

$$2\int_{\theta=0}^{2\pi}\int_{z=0}^{(\lambda_D{}^2-\rho_i{}^2)^{1/2}} d\theta dz\left[\frac{x}{\gamma}+\rho_i{}^2+z^2-2\rho_i\cos\theta\left(\frac{x}{\gamma}\right)^{1/2}\right]^{-1/2}$$

$$=2\int_{\theta=0}^{2\pi}\ln\left\{\left[\frac{x}{\gamma}-2\rho_i\cos\theta\left(\frac{x}{\gamma}\right)^{1/2}+\lambda_D{}^2\right]^{1/2}+(\lambda_D{}^2-\rho_i{}^2)^{1/2}\right\}d\theta-2\int_{\theta=0}^{2\pi}\ln\left[\frac{x}{\gamma}-2\rho_i\cos\theta\left(\frac{x}{\gamma}\right)^{1/2}+\rho_i{}^2\right]^{1/2}d\theta. \quad \text{(A14)}$$

First, consider the second term on the right-hand side:

$$\int_0^{2\pi}\ln[x+\gamma\rho_i{}^2-2\gamma^{1/2}x^{1/2}\rho_i\cos\theta]d\theta. \quad \text{(A15)}$$

Letting

$$y=e^{i\theta}, \quad \text{(A16)}$$

we obtain

$$\oint_{\text{(unit circle)}}\ln\left[x+\gamma\rho_i{}^2-\gamma^{1/2}x^{1/2}\rho_i\left(y+\frac{1}{y}\right)\right]\times\left(-i\frac{dy}{y}\right), \quad \text{(A17)}$$

or, with

$$a=x^{1/2},\quad b=\gamma^{1/2}\rho_i, \quad \text{(A18)}$$

alternately

$$-i\oint\ln(-ab)\frac{dy}{y}-i\oint\ln\left(y-\frac{a}{b}\right)\frac{dy}{y}-i\oint\ln\left(y-\frac{a}{b}\right)\frac{dy}{y}+i\oint\ln y\frac{dy}{y}. \quad \text{(A19)}$$

Using the contour illustrated in Fig. 3, we evaluate the complex integrals, obtaining

$$\begin{aligned}&2\pi\ln(-x)+iC, && x^{1/2}\gamma^{-1/2}\rho_i{}^{-1}>1, && C=\text{const},\\ &2\pi\ln(-\gamma\rho_i{}^2)+iC, && x^{1/2}\gamma^{-1/2}\rho_i{}^{-1}<1, && C=\text{const}.\end{aligned} \quad \text{(A20)}$$

The corresponding contribution to $\Delta(n,\rho_i)$ is

$$-2\pi\int_{x=\gamma\rho_i{}^2}^{\infty}\ln x\left[\frac{e^{-x}x^n}{n!}-\frac{e^{-x}x^{n-1}}{(n-1)!}\right]\left[\frac{-Ze^2}{2\pi\hbar v}\right]dx \quad \text{(A21)}$$

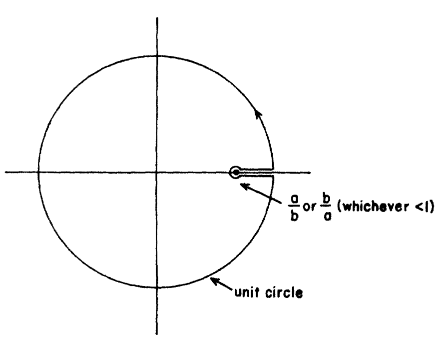

FIG. 3. The contour integral of Eq. (A19).

for $x^{1/2}/\gamma^{1/2}\rho_i>1$, and

$$-2\pi\int_{x=0}^{\gamma\rho_i{}^2}\ln(\gamma\rho_i{}^2)\left[\frac{e^{-x}x^n}{n!}-\frac{e^{-x}x^{n-1}}{(n-1)!}\right]\left[\frac{-Ze^2}{2\pi\hbar v}\right]dx \quad \text{(A22)}$$

for $x^{1/2}/\gamma^{1/2}\rho_i<1$.

The sum of contributions (A21) and (A22) can be rewritten in terms of the incomplete gamma function

$$\gamma(a,x)=\int_0^x e^{-t}t^{a-1}dt,\quad \Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}dt. \quad \text{(A23)}$$

Disregarding the factor $Ze^2/\hbar v$, one obtains

$$\ln(\gamma\rho_i{}^2)\left[\frac{\gamma(n+1,\gamma\rho_i{}^2)}{n!}-\frac{\gamma(n,\gamma\rho_i{}^2)}{(n-1)!}\right]+\int_{x=\gamma\rho_i{}^2}^{\infty}\left[\frac{e^{-x}x^n}{n!}-\frac{e^{-x}x^{n-1}}{(n-1)!}\right]\ln x dx. \quad \text{(A24)}$$

With the identity

$$\int e^{-x}x^{n-1}\ln x dx=e^{-x}x^{n-1}(x\ln x-x)-\int(n-1)x^{n-1}e^{-x}\ln x dx-\int e^{-x}x^n dx+\int e^{-x}x^n\ln x dx+\int(n-1)e^{-x}x^{n-1}dx, \quad \text{(A25)}$$

one obtains for the integral in (A24)

$$\int\left[\frac{e^{-x}x^n}{n!}-\frac{e^{-x}x^{n-1}}{(n-1)!}\right]\ln x dx=-\frac{1}{n!}e^{-x}x^{n-1}(x\ln x-x)\Big|_{\gamma\rho_i{}^2}^{\infty}+\frac{1}{n!}[\Gamma(n+1,x)-(n-1)\Gamma(n,x)]. \quad \text{(A26)}$$

Expression (A24) then becomes

$$\Gamma(n,x)/n!, \quad \text{(A27)}$$

or approximately,

$$\begin{matrix} 1/n, & \gamma\rho_i^2<n, \\ 0, & \gamma\rho_i^2>n, \end{matrix} \quad \text{for } n\ll 1. \tag{A28}$$

Note that

$$n\approx KT/\hbar\omega_c\geq 10^4T/H. \tag{A29}$$

In the classical limit, our approximation (A28) holds exactly.

To complete the evaluation of $\Delta(n,\rho_i)$ we must integrate:

$$-\frac{Ze^2}{\pi\hbar v_z}\int\int\left[\frac{e^{-x}x^n}{n!}-\frac{e^{-x}x^{n-1}}{(n-1)!}\right] \times\ln[(x-2x^{1/2}\gamma^{1/2}\rho_i\cos\theta+\gamma\lambda_D^2)^{1/2}+\gamma^{1/2}(\lambda_D^2-\rho_i^2)^{1/2}]dxd\theta. \tag{A30}$$

Since $n\gg 1$, $e^{-x}x^n$ is sharply peaked at $x=n$. We therefore obtain

$$-\frac{Ze^2}{\pi\hbar v_z}\int_{\theta=0}^{2\pi}\left\{\ln\left[\left(\frac{n}{\gamma}-2\frac{n^{1/2}}{\gamma^{1/2}}\rho_i\cos\theta+\lambda_D^2\right)^{1/2}+(\lambda_D^2-\rho_i^2)^{1/2}\right] -\ln\left[\left(\frac{n-1}{\gamma}-2\frac{(n-1)^{1/2}}{\gamma^{1/2}}\rho_i\cos\theta+\lambda_D^2\right)^{1/2}+(\lambda_D^2-\rho_i^2)^{1/2}\right]\right\}d\theta. \tag{A31}$$

Now,

$$\log[f(n)]-\log[f(n-1)]=f'(n)/f(n), \tag{A32}$$

and the expression (A31) becomes

$$-\frac{Ze^2}{\pi\hbar v_z}\int_0^{2\pi}\left[\frac{1}{\gamma}-\frac{\rho_i}{n^{1/2}\gamma^{1/2}}\cos\theta\right]\left[\left(\frac{n}{\gamma}-2\frac{n^{1/2}}{\gamma^{1/2}}\rho_i\cos\theta+\lambda_D^2\right)^{1/2}+(\lambda_D^2-\rho_i^2)^{1/2}\right]^{-1}\left[\frac{n}{\gamma}-2\frac{n^{1/2}}{\gamma^{1/2}}\rho_i\cos\theta+\lambda_D^2\right]^{1/2}d\theta. \tag{A33}$$

Since $n=\gamma r_L^2$, and since we assume $r_L\ll\lambda_D$, this expression may be expanded in a rapidly convergent power series in $\cos\theta$. The term in $\cos\theta$ vanishes upon integration. To order of the dominant factor in the coefficient for $\cos^2\theta$ we have

$$-\frac{Ze^2}{2\pi\hbar v_z}\int_0^{2\pi}\left\{[\gamma\lambda_D^2+\gamma\lambda_D(\lambda_D^2-\rho_i^2)^{1/2}]^{-1} -\cos^2\theta\left[\frac{2\rho_i^2}{\gamma[\lambda_D^4+2\lambda_D^3(\lambda_D^2-\rho_i^2)^{1/2}+\lambda_D^2(\lambda_D^2-\rho_i^2)]}+\frac{\rho_i^2(\lambda_D^2-\rho_i^2)^{1/2}}{\gamma\lambda_D[\lambda_D^4+2\lambda_D^3(\lambda_D^2-\rho_i^2)^{1/2}+\lambda_D^2(\lambda_D^2-\rho_i^2)]}\right]\right\}d\theta \tag{A34}$$

and

$$\Delta(n,\rho_i)=\frac{Ze^2}{\hbar vn}-\frac{Ze^2}{\hbar v}\left\{[\gamma\lambda_D^2+\gamma\lambda_D(\lambda_D^2-\rho_i^2)^{1/2}]^{-1}-\rho_i^2[\gamma(\lambda_D^4+2\lambda_D^3(\lambda_D^2-\rho_i^2)^{1/2}+\lambda_D^2(\lambda_D^2-\rho_i^2))]^{-1} -\rho_i^2(\lambda_D^2-\rho_i^2)^{1/2}[2\gamma\lambda_D(\lambda_D^4+2\lambda_D^3(\lambda_D^2-\rho_i^2)^{1/2}+\lambda_D^2(\lambda_D^2-\rho_i^2))]^{-1}\right\}\equiv\Delta_0 \tag{A35}$$

if $\gamma\rho_i^2<n$,

$$\Delta(n,\rho_i)=\Delta_0-Ze^2/\hbar vn \tag{A36}$$

if $\gamma\rho_i^2>n$. Equations (A35) and (A36) are the desired result.

APPENDIX B

The bremsstrahlung cross sections mentioned in Sec. 8 are obtained in the following way. The conductivity σ_ω (sec^{-1}) for frequencies between ω and $\omega+d\omega$, and the absorption coefficient per cm, κ_ω, are related by

$$\sigma_\omega=(c/4\pi)\kappa_\omega, \tag{B1}$$

with

$$\kappa_\omega=\frac{N_eN_i}{\omega^2}\frac{32\pi^2Z^2e^6}{3(2\pi)^{1/2}m^3c}\left(\frac{m}{KT}\right)^{3/2}\ln\left[\frac{4KT}{\gamma\hbar\omega}\right]. \tag{B2}$$

In Eq. (B2), $\gamma=1.78\cdots$ is Euler's constant. For temperatures of the order 10^5 °K and lower, the argument of the log term must be altered slightly.

The dc limit of the conductivity can be written in terms of a cross section $Q(\omega)$ with the following functional dependence:

$$\sigma_0\propto[Q(\omega\rightarrow 0)]^{-1}. \tag{B3}$$

On the other hand, the dependence of the conductivity for finite frequencies on the cross section is conventionally written as

$$\sigma_\omega\propto Q(\omega)/\omega^2. \tag{B4}$$

The consistency of Eqs. (B3) and (B4) can be verified by computing the limit $\omega\rightarrow 0$ of $Q(\omega)$.

We then have

$$Q(\omega)=\frac{N_i\pi^{3/2}Z^2e^6}{4(6)^{1/2}(KT)^2}\ln\left[\frac{4KT}{\gamma\hbar\omega_c}\right]. \tag{B5}$$